Available online at www.sciencedirect.com
ScienceDirect
Journal of Approximation Theory 186 (2014) 28–32
Journal of Approximation Theory
www.elsevier.com/locate/jat

Full length article

Two measures on Cantor sets

Gökalp Alpan, Alexander Goncharov*

Bilkent University, 06800, Ankara, Turkey

Received 22 April 2014; received in revised form 12 June 2014; accepted 9 July 2014
Available online 23 July 2014

Communicated by Vilmos Totik

Abstract

We give an example of Cantor-type set for which its equilibrium measure and the corresponding Hausdorff measure are mutually absolutely continuous. Also we show that these two measures are regular in the Stahl–Totik sense.

MSC: 30C85; 31A15; 28A78; 28A80

Keywords: Harmonic measures; Hausdorff measures; Cantor sets

1. Introduction

The relation between the α dimensional Hausdorff measure Λ_α and the harmonic measure ω on a finitely connected domain Ω is understood well. Due to Makarov [5], we know that, for a simply connected domain, $\dim \omega = 1$ where $\dim \omega := \inf\{\alpha : \omega \perp \Lambda_\alpha\}$. Pommerenke [9] gives a full characterization of parts of $\partial \Omega$ where ω is absolutely continuous or singular with respect to a linear Hausdorff measure. Later similar facts were obtained for finitely connected domains. In the infinitely connected case there are only particular results. Model example here is $\Omega = \overline{\mathbb{C}} \setminus K$ for a Cantor-type set K. For all such cases we have $\Lambda_{\alpha_K} \perp \omega$ on K, because of the strict inequality $\dim \omega < \alpha_K$ (see, e.g. [1,6,7,12,14]), where α_K stands for the Hausdorff dimension of K.

* Corresponding author.
E-mail addresses: gokalp@fen.bilkent.edu.tr (G. Alpan), goncha@fen.bilkent.edu.tr (A. Goncharov).

http://dx.doi.org/10.1016/j.jat.2014.07.003
0021-9045/

These results motivate the problem to find a Cantor set for which its harmonic measure and the corresponding Hausdorff measure are not mutually singular.

Recall that, for a dimension function h, a set $E \subset \mathbb{C}$ is an h-set if $0 < \Lambda_h(E) < \infty$ where Λ_h is the Hausdorff measure corresponding to the function h. We consider Cantor-type sets $K(\gamma)$ introduced in [3]. In Section 2 we present a function h that makes $K(\gamma)$ an h-set. In Section 3 we show that Λ_h and ω are mutually absolutely continuous for $K(\gamma)$. In the last section we prove that these two measures are regular in the Stahl–Totik sense.

We will denote by log the natural logarithm, and $Cap(\cdot)$ stands for the logarithmic capacity.

2. Dimension function of $K(\gamma)$

A function $h : \mathbb{R}_+ \to \mathbb{R}_+$ is called a dimension function if it is increasing, continuous and $h(0) = 0$. Given set $E \subset \mathbb{C}$, its h-Hausdorff measure is defined as

$$\Lambda_h(E) = \lim_{\delta\to 0} \inf \left\{ \sum h(r_j) : E \subset \bigcup B(z_j, r_j) \text{ with } r_j \le \delta \right\}, \tag{2.1}$$

where $B(z, r)$ is the open ball of radius r centered at z.

For the convenience of the reader we repeat the relevant material from [3]. Given sequence $\gamma = (\gamma_s)_{s=1}^{\infty}$ with $0 < \gamma_s \le \frac{1}{32}$, let $r_0 = 1$ and $r_s = \gamma_s r_{s-1}^2$ for $s \in \mathbb{N}$. Define $P_2(x) = x(x-1)$ and $P_{2^{s+1}} = P_{2^s} \cdot (P_{2^s} + r_s)$ for $s \in \mathbb{N}$. Consider the set

$$E_s := \{x \in \mathbb{R} : \ P_{2^{s+1}}(x) \le 0\} = \cup_{j=1}^{2^s} I_{j,s}.$$

The sth level *basic* intervals $I_{j,s}$ with lengths $l_{j,s}$ are disjoint and $\max_{1\le j\le 2^s} l_{j,s} \to 0$ as $s \to \infty$. Since $E_{s+1} \subset E_s$, we have a Cantor-type set $K(\gamma) := \cap_{s=0}^{\infty} E_s$. The set $K(\gamma)$ is non-polar if and only if $\sum_{s=1}^{\infty} 2^{-s} \log \frac{1}{\gamma_s} < \infty$. In this paper we make the assumption

$$\sum_{s=1}^{\infty} \gamma_s < \infty. \tag{2.2}$$

Let $M := 1 + \exp\left(16 \sum_{s=1}^{\infty} \gamma_s\right)$, so $M > 2$, and $\delta_s := \gamma_1\gamma_2 \ldots \gamma_s$. By Lemma 6 in [3],

$$\delta_s < l_{j,s} < M \cdot \delta_s \quad \text{for } 1 \le j \le 2^s. \tag{2.3}$$

We construct a dimension function for $K(\gamma)$, following Nevanlinna [8]. Let $\eta(\delta_s) = s$ for $s \in \mathbb{Z}_+$ with $\delta_0 := 1$. We define $\eta(t)$ for (δ_{s+1}, δ_s) by

$$\eta(t) = s + \frac{\log \frac{\delta_s}{t}}{\log \frac{\delta_s}{\delta_{s+1}}}.$$

This makes η continuous and monotonically decreasing on $(0, 1]$. In addition, we have $\lim_{t\to 0} \eta(t) = \infty$. Also observe that, for the derivative of η on (δ_{s+1}, δ_s), we have

$$\frac{d\eta}{dt} = \frac{-1}{t \log \frac{1}{\gamma_{s+1}}} \ge \frac{-1}{t \log 32} \quad \text{and} \quad \frac{d\eta}{d \log t} \ge \frac{-1}{\log 32}.$$

Define $h(t) = 2^{-\eta(t)}$ for $0 < t \le 1$ and $h(t) = 1$ for $t > 1$. Then h is a dimension function with $h(\delta_s) = 2^{-s}$ and

$$\frac{d \log h}{d \log t} < \frac{\log 2}{\log 32} < 1.$$

Therefore if $m > 1$ and $r \le 1$ we get the following inequality:

$$\log \frac{h(r)}{h\left(\frac{r}{m}\right)} < \int_{r/m}^{r} d\log t = \log m.$$

Finally, we obtain

$$h(r) < m \cdot h\left(\frac{r}{m}\right) \quad \text{for } m > 1 \text{ and } 0 < r \le 1. \tag{2.4}$$

Let us show that $K(\gamma)$ is an h-set for the given function h.

Theorem 2.1. *Let γ satisfy (2.2). Then $1/8 \le \Lambda_h(K(\gamma)) \le M/2$.*

Proof. First, observe that, by (2.3), for each $s \in \mathbb{N}$ the set $K(\gamma)$ can be covered by 2^s intervals of length $M \cdot \delta_s$. Since $M/2 > 1$, we have by (2.4),

$$\Lambda_h(K(\gamma)) \le \limsup_{s\to\infty}(2^s \cdot h(M/2 \cdot \delta_s)) \le \limsup_{s\to\infty}(2^s \cdot M/2 \cdot h(\delta_s)) = M/2.$$

We proceed to show the lower bound. Let (J_ν) be an open cover of $K(\gamma)$. Then, by compactness, there are finitely many intervals $(J_\nu)_{\nu=1}^m$ that cover $K(\gamma)$. Since $K(\gamma)$ is totally disconnected, we can assume that these intervals are disjoint. Each J_ν contains a closed subinterval $J'_\nu = [a_\nu, b_\nu]$ whose endpoints belong to $K(\gamma)$ and covers all points of $K(\gamma)$ in J_ν. Since the intervals $(J'_\nu)_{\nu=1}^m$ are disjoint, all a_ν, b_ν are endpoints of some basic intervals. Let n be the minimal number such that all $(a_\nu)_{\nu=1}^m$, $(b_\nu)_{\nu=1}^m$ are the endpoints of nth level. Thus, each $I_{j,n}$ for $1 \le j \le 2^n$ is contained in some J'_ν. Let N_ν be the number of nth level intervals in J'_ν. Clearly, $\sum_{\nu=1}^m N_\nu = 2^n$.

For a fixed $\nu \in \{1, 2, \ldots, m\}$, let q_ν be the smallest number such that J'_ν contains at least one basic interval I_{j,q_ν}. Clearly, $q_\nu \le n$ and $l_{j,q_\nu} \le d_\nu$ where d_ν is the length of J_ν. Therefore, by (2.3),

$$h(d_\nu) \ge h(l_{j,q_\nu}) \ge h(\delta_{q_\nu}) = 2^{-q_\nu}.$$

Let us cover J'_ν by the smallest set G_ν which is a finite union of adjacent intervals of the level q_ν. Observe that G_ν consists of at least one and at most four such intervals. Each interval of the q_νth level contains 2^{n-q_ν} subintervals of the nth level. This gives at most $2^{n-q_\nu+2}$ intervals of level n in the set G_ν. Hence

$$N_\nu \le 2^{n-q_\nu+2}.$$

Therefore,

$$\sum_{\nu=1}^m h(d_\nu) \ge \sum_{\nu=1}^m 2^{-q_\nu} \ge 2^{-n-2} \sum_{\nu=1}^m N_\nu = 1/4.$$

Since $h(d) < 2 \cdot h(d/2)$ from (2.4), finally we obtain the desired bound. $\square$

Similar arguments apply to the case of a part of $K(\gamma)$ on any basic interval.

Corollary 2.2. *Let γ satisfy (2.2). Then $2^{-s-3} \le \Lambda_h(K(\gamma) \cap I_{j,s}) \le M \cdot 2^{-s-1}$ for each $s \in \mathbb{N}$ and $1 \le j \le 2^s$.*

Remark. A set E is called *dimensional* if there is at least one dimension function h that makes E an $h-$set. It should be noted that not all sets are dimensional. If we replace the condition $h(0) = 0$ by $h(0) \ge 0$, then any sequence gives a trivial example of a dimensionless set.

Best in [2] presented an example of a dimensionless Cantor set provided $h(0) = 0$. The author considered dimension functions with the additional condition of concavity, but did not used it in his construction.

3. Harmonic measure and Hausdorff measure for $K(\gamma)$

Suppose we are given a non-polar compact set K that coincides with its exterior boundary. Then for the equilibrium measure μ_K on K we have the representation $\mu_K(\cdot) = \omega(\infty, \cdot, \overline{\mathbb{C}} \setminus K)$ in terms of the value of the harmonic measure at infinity (see e.g. [10], T.4.3.14). Moreover, since measures $\omega(z_1, \cdot, \overline{\mathbb{C}} \setminus K)$ and $\omega(z_2, \cdot, \overline{\mathbb{C}} \setminus K)$ are mutually absolutely continuous (see e.g. [10] Cor. 4.3.5), our main result is valid even if, instead of $\mu_{K(\gamma)}$, we take the measure corresponding to the value of the harmonic measure at any other point.

The set $K(\gamma)$ is weakly equilibrium in the following sense. Given $s \in \mathbb{N}$, we uniformly distribute the mass 2^{-s} on each $I_{j,s}$ for $1 \le j \le 2^{-s}$. Let us denote by λ_s the normalized in this sense Lebesgue measure on E_s, so $d\lambda_s = (2^s l_{j,s})^{-1} dt$ on $I_{j,s}$.

Theorem 3.1 (*[3], T.4*). *Suppose $K(\gamma)$ is not polar. Then λ_s is weak star convergent to the equilibrium measure $\mu_{K(\gamma)}$.*

Corollary 3.2. *Suppose $K(\gamma)$ is not polar. Then $\mu_{K(\gamma)}(I_{j,s}) = 2^{-s}$ for each $s \in \mathbb{N}$ and $1 \le j \le 2^s$.*

Proof. Indeed, the characteristic function $\chi_{I_{j,s}}$ is continuous on E_n for $n \ge s$, where E_n is given in the construction of $K(\gamma)$. Therefore, $\mu_{K(\gamma)}(I_{j,s}) = \int \chi_{I_{j,s}} \, d\mu_{K(\gamma)} = \lim_{n\to\infty} \int \chi_{I_{j,s}} \, d\lambda_n = 2^{-s}$. □

In order to compare measures on Cantor-type sets, we use a standard technique.

Lemma 3.3. *Suppose μ and ν are finite Borel measures on a Cantor-type set K. Let $C_1 \, \mu(I) \le \nu(I) \le C_2 \, \mu(I)$ for each basic interval I with some positive constants C_1, C_2. Then $C_1 \, \mu(E) \le \nu(E) \le C_2 \, \mu(E)$ for each Borel set E.*

By assumption, the measures μ, ν are comparable with the same constants on any interval, then on open sets and, by regularity, on Borel sets.

Corollaries 2.2 and 3.2 with Lemma 3.3 imply the next theorem, where (and below) by Λ_h we mean restricted to the compact set $K(\gamma)$ the Hausdorff measure corresponding to the constructed function h.

Theorem 3.4. *Let γ satisfy (2.2) and $K(\gamma)$ be non-polar. Then measures $\mu_{K(\gamma)}$ and Λ_h are mutually absolutely continuous.*

4. Regularity of $\mu_{K(\gamma)}$ and Λ_h in the Stahl–Totik sense

One of active directions of the theory of general orthogonal polynomials is the exploration of the case of non-discrete measures that are singular with respect to the Lebesgue measure. Important class of *regular in the Stahl–Totik sense* measures was introduced in [11] in the following way. Let μ be a finite Borel measure with compact support S_μ on $\mathbb{C}$. Then we can uniquely define a sequence of orthonormal polynomials $p_n(\mu; z) = a_n z^n + \cdots$ with a positive leading coefficient a_n. By definition, $\mu \in \mathbf{Reg}$ if $\lim_{n\to\infty} a_n^{-\frac{1}{n}} = Cap(S_\mu)$. One of sufficient conditions of regularity was suggested in [11] by means of the set $A_\mu = \{z \in S_\mu : \limsup_{r\to 0^+} \frac{\log 1/\mu(\overline{B(z,r)})}{\log 1/r} < \infty\}$.

Theorem 4.1 (*T.4.2.1 in [11]*). *If $Cap(A_\mu) = Cap(S_\mu)$ then $\mu \in$* **Reg**.

Let us show that, in our case, $A_\mu = S_\mu$ for both measures $\mu_{K(\gamma)}$ and Λ_h. Compare this with [13].

Theorem 4.2. *Let $K(\gamma)$ satisfy the conditions of Theorem 3.4. Then $\mu_{K(\gamma)}$ and Λ_h are regular in the Stahl–Totik sense.*

Proof. Since $\Lambda_h(E) \geq \mu_{K(\gamma)(E)}/8$ for any Borel subset E of $K(\gamma)$, we only check the equilibrium measure. Let $z \in K(\gamma)$ and $r > 0$ be given. Fix s such that $z \in I_{i,s} \subset I_{j,s-1}$ with $l_{i,s} \leq r < l_{j,s-1}$. Then $I_{i,s} \subset \overline{B(z,r)}$. By Corollary 3.2, $\mu(\overline{B(z,r)}) \geq \mu(I_{i,s}) = 2^{-s}$. On the other hand, $r < M\,\delta_{s-1} \leq M\,32^{1-s}$ as $\gamma_k \leq 1/32$. Since $s \to \infty$ as $r \to 0^+$, we see that

$$\limsup_{r\to 0^+} \frac{\log 1/\mu(\overline{B(z,r)})}{\log 1/r} \leq 1/5,$$

which completes the proof. □

It should be mentioned that regularity of a certain class of singular continuous measures, including the Cantor–Lebesgue measure for the classical ternary set, was proven in the recent paper [4].

References

[1] A. Batakis, Harmonic measure of some Cantor type sets, Ann. Acad. Sci. Fenn. 21 (1996) 27–54.
[2] E. Best, A closed dimensionless linear set, Proc. Edinburgh Mat. Soc. 6 (2) (1939) 105–108.
[3] A. Goncharov, Weakly equilibrium cantor-type sets, Potential Anal. 40 (2) (2014) 143–161.
[4] H. Krüger, B. Simon, Cantor polynomials and some related classes of OPRL, J. Approx. Theory (2014) http://dx.doi.org/10.1016/j.jat.2014.04.003.
[5] N.G. Makarov, On the distortion of boundary sets under conformal mappings, Proc. Lond. Math. Soc. 51 (1985) 369–384.
[6] N.G. Makarov, Fine structure of harmonic measure, Algebra i Analiz 10 (2) (1998) 1–62.
[7] N.G. Makarov, A. Volberg, On the harmonic measure of discontinuous fractals, LOMI Preprints, E-6-86, Steklov Mathematical Institute, Leningrad Department, 1986.
[8] R. Nevanlinna, Analytic Functions, Springer-Verlag, 1970.
[9] Ch. Pommerenke, On conformal mapping and linear measure, J. Anal. Math. 46 (1986) 231–238.
[10] T. Ransford, Potential theory in the complex plane, Cambridge University Press, 1995.
[11] H. Stahl, V. Totik, General Orthogonal Polynomials, Cambridge University Press, 1992.
[12] A. Volberg, On the dimension of harmonic measure of Cantor repellors, Michigan Math. J. 40 (2) (1993) 239–258.
[13] H. Widom, Polynomials associated with measures in the complex plane, J. Math. Mech. 16 (1967) 997–1013.
[14] A. Zdunik, Harmonic measure on the Julia set for polynomial-like maps, Invent. Math. 128 (2) (1997) 303–327.